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Efficient Class of Optimized Coning Compensation Algorithms

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Various algorithms are examined for integrating the noncommutivity rate equation that is the basis for the coning correction applied in strapdown inertial reference systems. A concept for improving the computational accuracy and efficiency of strapdown coning correction algorithms is described and applied to the development of six new algorithms, each possessing optimal accuracy characteristics and leading to minimum computational throughput requirements. The accuracies of the six new algorithms are compared to those associated with four previously known algorithms in both pure coning and benign angular rate environments. It is shown that, for all algorithms, the error incurred in a benign environment is negligibly small, even for very severe maneuvers, and that algorithm optimization based strictly on a consideration of its accuracy characteristics in a pure coning environment is justified. An algorithm simplification concept employed that is similarly based solely on the properties of the algorithm in a pure coning environment is also shown to be justified.

Nomenclature

a, b	= amplitudes of the angular oscillations in two orthogonal axes of the body
I, J	= unit vectors along the two body axes about which the oscillations are occurring
N	= the number of subminor sensor-data intervals in a minor computational interval
$\Delta\theta_m(i)$	= incremental angle vector over the ith subminor interval of the mth minor interval
$\delta\hat{\theta}_c(m)$	= algorithmic approximation to the coning integral over the mth minor computational interval
Ω	= frequency associated with the angular oscillations
ω	= angular velocity vector expressed with coordinates in the body frame

Introduction

THE generalized solution to the attitude reference problem, as given by Bortz,[1] establishes the complete theoretical basis for the attitude matrix computations carried out in strapdown inertial systems. The orientation vector differential equation, derived in general form,[1] defines the rate of change of the orientation vector as the sum of the inertially measured angular rate vector and a computationally determined noncommutivity rate vector, the latter arising as a consequence of the noncommutivity of finite rotations. The numerical integration of the noncommutivity rate equation is an important factor in the accuracy achievable in strapdown systems and establishes the basis for the coning compensation algorithms used to prevent attitude error buildup in highly dynamic angular rate environments.

A concept for optimizing coning compensation algorithms for strapdown inertial systems was first introduced and applied by Miller,[2] with subsequent applications of the technique given in Refs. 3–5. The optimization procedure consists of selecting the coefficients of a given higher-order coning compensation algorithm in a manner that leads to minimum error in a pure coning angular rate environment. The optimization is achieved, however, at the expense of slightly compromised algorithm performance in a benign dynamic environment, such as experienced during periods of vehicle maneuvering but, as shown in Ref. 3, this type of error constitutes a higher-order effect and, as will be seen, is ignorable even for severe vehicle maneuvers. Reference 3 also shows that, although assuming a pure coning angular rate environment, the algorithm optimization procedure is applicable to generalized angular motions such as are experienced in a sustained random angular vibration environment and, therefore, constitutes a completely general approach for maximizing the performance of higher-order coning compensation algorithms.

A concept for further enhancing the accuracy and efficiency of coning compensation algorithms was introduced by Jiang and Lin[5] and applied to the modification of two coning compensation algorithms, due to Miller[2] and Jordan.[6] The modified algorithm consists of the weighted sum of the terms constituting the nominal coning compensation algorithm and a suitably chosen additional term, with the coefficients of the resultant augmented algorithm being optimized using Miller's technique.

An alternate coning algorithm enhancement concept is possible that leads to optimal accuracy characteristics and minimum computational throughput requirements and also to an algorithm structure that lends itself naturally to achieving a desired balance between performance and computational load. This alternate concept for enhancing the performance of coning compensation algorithms is the subject of the development to follow. Six new coning compensation algorithms are defined for the four cases in which the coning correction is computed over successive periods of time spanning two, three, four, or five sensor-data intervals. For each, the errors incurred in both pure coning and benign environments are defined and compared with the accuracies associated with other known algorithms.

Coning Correction

The coning correction $\Delta\theta_c(n)$ over the attitude update interval from t_{n-1} to t_n is defined as the integral of the noncommutivity rate vector over the interval or, explicitly,

$$\Delta\theta_c(n) = \frac{1}{2}\int_{t_{n-1}}^{t_n} \alpha(t, t_{n-1}) \times \omega \, dt \tag{1}$$

where

$$\alpha(t, t_{n-1}) = \int_{t_{n-1}}^{t} \omega \, dt$$

and ω is the angular rate vector, the components of which are sensed by a set of three gyros having orthogonal input axes. Equation (1) assumes the commonly accepted simplified form of the noncommutivity rate vector, as opposed to the generalized form of Ref. 1.

The coning correction defined by Eq. (1) may be carried out by dividing the major computational interval over which the attitude

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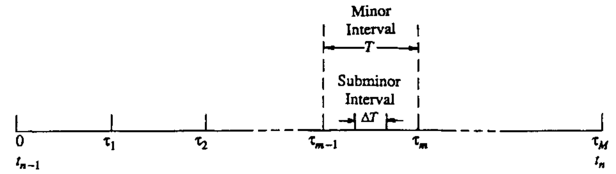

Fig. 1 Intervals associated with computation of coning correction.

matrix is to be updated into a number of minor intervals, each in turn being divided into a number of subminor sensor-data intervals, as illustrated in Fig. 1. The coning correction over a major computational interval can be expressed as the sum of the contributions from the M minor intervals, each computed according to Eq. (1), as follows:

$$\Delta\theta_c(n) = \sum_{m=1}^{M} \frac{1}{2}\int_{\tau_{m-1}}^{\tau_m} \alpha(t_{n-1}+\tau, t_{n-1}) \times \omega \,\mathrm{d}\tau$$

$$= \frac{1}{2}\sum_{m=2}^{M} \alpha_{m-1} \times \Delta\theta_m + \frac{1}{2}\sum_{m=1}^{M}\int_{\tau_{m-1}}^{\tau_m} \alpha(\tau, \tau_{m-1}) \times \omega \,\mathrm{d}\tau \quad (2)$$

in which the following definitions apply:

$$\alpha(\tau, \tau_{m-1}) = \int_{\tau_{m-1}}^{\tau} \omega \,\mathrm{d}\tau$$

$$\Delta\theta_m = \int_{\tau_{m-1}}^{\tau_m} \omega \,\mathrm{d}\tau$$

$$\alpha_{m-1} = \int_{t_{n-1}}^{t_{n-1}+\tau_{m-1}} \omega \,\mathrm{d}t = \sum_{i=1}^{m-1} \Delta\theta_i$$

Define the coning integral over a minor computational interval from τ_{m-1} to τ_m as

$$\delta\theta_c(m) = \frac{1}{2}\int_{\tau_{m-1}}^{\tau_m} \alpha(\tau, \tau_{m-1}) \times \omega \,\mathrm{d}\tau \quad (3)$$

the accurate numerical integration of which constitutes the central issue in the coning compensation algorithm design problem, since it alone contributes error to the coning correction over a major attitude update interval, the first term in Eq. (2) being inherently error free. A number of algorithms for numerical evaluation of the coning integral, representing varying degrees of complexity, are given in the following.

Coning Algorithms and Accuracies

A generalized algorithmic approximation to the coning integral defined by Eq. (3) consists of the sum of all possible cross products formed from the incremental angle vectors for the N subminor sensor-data intervals making up the minor computational interval, a total of $N(N-1)/2$ terms. The algorithmic approximation is defined by

$$\delta\hat{\theta}_c(m) = \sum_{i=1}^{N-1}\sum_{j=i+1}^{N} b_{ij}\Delta\theta_m(i) \times \Delta\theta_m(j) \quad (4)$$

where the b_{ij} are constants.

In a pure coning angular rate environment, the contribution to the coning integral of incremental-angle vector cross products having equal time separations is the same regardless of the absolute times associated with the two intervals. Taking advantage of this property allows the generalized form for the coning integral algorithm to be simplified as

$$\delta\hat{\theta}_c(m) = \left[\sum_{i=1}^{N-1} b_i \Delta\theta_m(i)\right] \times \Delta\theta_m(N) \quad (5)$$

with the b_i being constant coefficients.

The computational throughput requirements associated with the simplified algorithm structure defined by Eq. (5) are significantly reduced, with only one cross product appearing explicitly in the equation, and the equivalent of a half-cross product accounted for implicitly in each term within the brackets, resulting in a total computational load of $(N+1)/2$ cross-product equivalents. The number of cross-product equivalents for the generalized algorithm structure given by Eq. (4) is found in a similar manner to be $3N(N-1)/4$. Therefore, for $N = 3$, the computational load is determined for the simplified algorithm to be 4/9 of that associated with the generalized algorithm. Similarly, for $N = 4$, the computational load is determined to be only 5/18 of that associated with the generalized algorithm and, for $N = 5$, the load is found to be only 1/5 that of the generalized algorithm.

The penalty associated with the simplified form of the coning algorithm is increased error in a benign environment. As will be shown, however, this error is at an ignorable level, even for severe vehicle maneuvers; furthermore, the resultant attitude error growth persists only for the duration of the maneuver, whereas any mean coning compensation error induced in a sustained coning angular rate environment will produce a steady attitude error growth, which favors optimizing the coning compensation algorithm for coning inputs and accepting the negligibly small penalty associated with the benign environment.

The result given by Eq. (5) is not unique since, in fact, the following alternate form produces the same result in a pure coning environment:

$$\delta\hat{\theta}_c(m) = \left[\sum_{i=2}^{N} b_i \Delta\theta_m(i)\right] \times \Delta\theta_m(1) \quad (6)$$

The first form is preferable, however, for the purpose of defining an enhanced coning compensation algorithm structure, since the following computationally efficient augmented form is its natural extension:

$$\delta\hat{\theta}_c(m) = \left[\sum_{i=1}^{p} a_i \Delta\theta_{m-1}(N-i+1) + \sum_{i=1}^{N-1} b_i \Delta\theta_m(i)\right] \times \Delta\theta_m(N) \quad (7)$$

where p is the number of successive subminor intervals utilized from the previous minor computational interval (ending at the Nth), with the a_i and b_i being constants. The augmented form defined by Eq. (7) generates incremental-angle cross products having separations of 1 to $N+p$ sensor-data intervals. It represents the simplest form possible for generating this many distinct sensor-data cross products and, as will be shown, the greater the number of distinct cross products utilized in an algorithm, the greater is the accuracy achieved.

Ten algorithms for numerical solution of the coning integral are given next, six of which are based on the enhanced algorithm structure defined by Eq. (7). Of the remaining four algorithms, three utilize the simplification concept implicit in Eq. (5). The coefficients of all algorithms have been optimized using the criterion that the coning compensation error is to be minimized in a pure coning angular rate environment. The ten algorithms, predicated on either two, three, four, or five subminor sensor-data intervals spanning a minor computational interval, are defined as follows.

Two-Interval Algorithms

A two-interval algorithm approximates the coning integral over successive minor computational intervals using sensor data from the two contiguous subminor computational intervals making up the minor interval. Two two-interval coning compensation algorithms are defined as follows:

Algorithm 1

$$\delta\hat{\theta}_c(m) = k_1\Delta\theta_m(1) \times \Delta\theta_m(2) + k_2\theta_{m-1} \times \theta_m$$

where

$$\theta_m = \Delta\theta_m(1) + \Delta\theta_m(2)$$

and

$$k_1 = 32/45, \qquad k_2 = -1/180$$

Algorithm 2

$$\delta\hat{\boldsymbol{\theta}}_c(m) = [k_1 \Delta\boldsymbol{\theta}_{m-1}(2) + k_2 \Delta\boldsymbol{\theta}_m(1)] \times \Delta\boldsymbol{\theta}_m(2)$$

where

$$k_1 = -1/30, \qquad k_2 = 11/15$$

Algorithm 1, which is due to Jiang and Lin,[5] represents an enhanced version of an algorithm due to Jordan,[6] the enhancement resulting from the addition of the term involving the cross product of the incremental-angle vector over the present minor interval with that over the previous minor interval.

Algorithm 2, which is a new algorithm, is also an enhanced version of the two-interval Jordan algorithm; the enhancement resulting in this case from the inclusion of sensor data from the second subminor interval of the previous minor interval, utilizing the algorithm structure defined by Eq. (7). (This algorithm also appears as Algorithm H in Ref. 3, where it was applied to the leftover minor interval, consisting of two subminor intervals, in a three-interval algorithm.)

Three-Interval Algorithms

A three-interval algorithm approximates the coning integral using sensor data from the three contiguous subminor computational intervals making up the minor computational interval. Four three-interval coning compensation algorithms are defined as follows:

Algorithm 3

$$\delta\hat{\boldsymbol{\theta}}_c(m) = [k_1 \Delta\boldsymbol{\theta}_m(1) + k_2 \Delta\boldsymbol{\theta}_m(2)] \times \Delta\boldsymbol{\theta}_m(3)$$

where

$$k_1 = 9/20, \qquad k_2 = 27/20$$

Algorithm 4

$$\delta\hat{\boldsymbol{\theta}}_c(m) = [k_1 \Delta\boldsymbol{\theta}_m(1) + k_2 \Delta\boldsymbol{\theta}_m(2)] \times \Delta\boldsymbol{\theta}_m(3) + k_3 \boldsymbol{\theta}_{m-1} \times \boldsymbol{\theta}_m$$

where

$$\boldsymbol{\theta}_m = \Delta\boldsymbol{\theta}_m(1) + \Delta\boldsymbol{\theta}_m(2) + \Delta\boldsymbol{\theta}_m(3)$$

and

$$k_1 = 243/560, \qquad k_2 = 1539/1120, \qquad k_3 = 1/3360$$

Algorithm 5

$$\delta\hat{\boldsymbol{\theta}}_c(m) = [k_1 \Delta\boldsymbol{\theta}_{m-1}(3) + k_2 \Delta\boldsymbol{\theta}_m(1) + k_3 \Delta\boldsymbol{\theta}_m(2)] \times \Delta\boldsymbol{\theta}_m(3)$$

where

$$k_1 = 3/280, \qquad k_2 = 57/140, \qquad k_3 = 393/280$$

Algorithm 6

$$\delta\hat{\boldsymbol{\theta}}_c(m) = [k_1 \Delta\boldsymbol{\theta}_{m-1}(2) + k_2 \Delta\boldsymbol{\theta}_{m-1}(3) + k_3 \Delta\boldsymbol{\theta}_m(1) + k_4 \Delta\boldsymbol{\theta}_m(2)] \times \Delta\boldsymbol{\theta}_m(3)$$

where

$$k_1 = -1/420, \qquad k_2 = 1/40, \qquad k_3 = 157/420, \qquad k_4 = 1207/840$$

Algorithm 3 is a modified version of an algorithm due to Miller.[2] It appears as Algorithm F in Ref. 3, and represents a more computationally efficient form of Miller's algorithm based on the simplified algorithm structure defined by Eq. (5).

Algorithm 4 is a specialized form of an algorithm due to Jiang and Lin,[5] which in turn, is an enhanced version of Miller's three-interval algorithm.[2] The enhancement results from the addition of the term involving the cross product of the incremental-angle vector over the present minor interval with that over the previous minor interval. The specialization comes about by employing the simplified form defined by Eq. (5), rather than the generalized form defined by Eq. (3).

Algorithm 5, which is a new algorithm, is also an enhanced version of Miller's three-interval algorithm, the enhancement resulting from the incorporation of sensor data from the last subminor interval of the previous minor interval, utilizing the augmented algorithm structure defined by Eq. (7).

Algorithm 6, also a new algorithm, represents a second enhanced version of Miller's three-interval algorithm, the enhancement resulting in this case from the utilization of sensor data from the last two subminor intervals of the previous minor interval. It demonstrates the flexibility of the augmented algorithm structure to incorporate data from as many subminor intervals of the previous minor interval as desired.

Four-Interval Algorithms

A four-interval algorithm approximates the coning integral using sensor data from the four contiguous subminor computational intervals constituting the minor computational interval. Two four-interval coning compensation algorithms are defined as follows:

Algorithm 7

$$\delta\hat{\boldsymbol{\theta}}_c(m) = [k_1 \Delta\boldsymbol{\theta}_m(1) + k_2 \Delta\boldsymbol{\theta}_m(2) + k_3 \Delta\boldsymbol{\theta}_m(3)] \times \Delta\boldsymbol{\theta}_m(4)$$

where

$$k_1 = 54/105, \qquad k_2 = 92/105, \qquad k_3 = 214/105$$

Algorithm 8

$$\delta\hat{\boldsymbol{\theta}}_c(m) = [k_1 \Delta\boldsymbol{\theta}_{m-1}(4) + k_2 \Delta\boldsymbol{\theta}_m(1) + k_3 \Delta\boldsymbol{\theta}_m(2) + k_4 \Delta\boldsymbol{\theta}_m(3)] \times \Delta\boldsymbol{\theta}_m(4)$$

where

$$k_1 = -1/315, \qquad k_2 = 168/315, \qquad k_3 = 262/315, \qquad k_4 = 656/315$$

Algorithm 7 is a specialized form of an algorithm due to Lee et al.[4] The specialization comes about by employing the simplified form defined by Eq. (5), as opposed to the generalized form defined by Eq. (3).

Algorithm 8, which is a new algorithm, is an enhanced four-interval algorithm that utilizes the algorithm structure defined by Eq. (7), with sensor data from the fourth subminor interval of the previous minor interval being incorporated in the computation of the coning integral over the present minor interval.

Five-Interval Algorithms

A five-interval algorithm approximates the coning integral using sensor data from the five contiguous subminor computational intervals that constitute the minor computational interval. Two five-interval coning compensation algorithms are defined as follows:

Algorithm 9

$$\delta\hat{\boldsymbol{\theta}}_c(m) = [k_1 \Delta\boldsymbol{\theta}_m(1) + k_2 \Delta\boldsymbol{\theta}_m(2) + k_3 \Delta\boldsymbol{\theta}_m(3) + k_4 \Delta\boldsymbol{\theta}_m(4)] \times \Delta\boldsymbol{\theta}_m(5)$$

where

$$k_1 = 125/252, \qquad k_2 = 25/24, \qquad k_3 = 325/252, \qquad k_4 = 1375/504$$

Algorithm 10

$$\delta\hat{\boldsymbol{\theta}}_c(m) = [k_1 \Delta\boldsymbol{\theta}_{m-1}(5) + k_2 \Delta\boldsymbol{\theta}_m(1) + k_3 \Delta\boldsymbol{\theta}_m(2) + k_4 \Delta\boldsymbol{\theta}_m(3) + k_5 \Delta\boldsymbol{\theta}_m(4)] \times \Delta\boldsymbol{\theta}_m(5)$$

Table 1 Coning algorithm accuracies in a pure coning environment

	Coning compensation algorithms									
	Two interval		Three interval				Four interval		Five interval	
Ratio[a]	1	2	3	4	5	6	7	8	9	10
1	0.100 e+3[b]	0.100 e+3	0.256 e+2	0.223 e+2	0.176 e+2	0.123 e+2	0.299 e+1	0.137 e+1	0.203	0.649 e−1
2	0.946 e+1	0.663 e+1	0.761	0.318	0.170	0.388 e−1	0.194 e−1	0.259 e−2	0.302 e−3	0.271 e−4
3	0.122 e+1	0.761	0.752 e−1	0.162 e−1	0.784 e−2	0.835 e−3	0.830 e−3	0.499 e−4	0.634 e−5	0.738 e−6
4	0.249	0.149	0.139 e−1	0.178 e−2	0.832 e−3	0.505 e−4	0.847 e−4	0.248 e−5	0.816 e−6	0.355 e−6
5	0.697 e−1	0.407 e−1	0.373 e−2	0.313 e−3	0.143 e−3	0.549 e−5	0.137 e−4	−0.777 e−8	0.379 e−6	0.242 e−6
6	0.242 e−1	0.140 e−1	0.126 e−2	0.745 e−4	0.338 e−4	0.826 e−6	0.278 e−5	−0.184 e−6	0.256 e−6	0.175 e−6
7	0.978 e−2	0.561 e−2	0.503 e−3	0.221 e−4	0.988 e−5	0.127 e−6	0.514 e−6	−0.166 e−6	0.191 e−6	0.132 e−6
8	0.445 e−2	0.254 e−2	0.227 e−3	0.766 e−5	0.338 e−5	−0.320 e−8	−0.336 e−7	−0.134 e−6	0.148 e−6	0.102 e−6
9	0.222 e−2	0.126 e−2	0.113 e−3	0.302 e−5	0.130 e−5	−0.279 e−7	−0.167 e−6	−0.108 e−6	0.118 e−6	0.818 e−7
10	0.118 e−2	0.673 e−3	0.600 e−4	0.131 e−5	0.545 e−6	−0.301 e−7	−0.188 e−6	−0.883 e−7	0.967 e−7	0.668 e−7

[a]Ratio of minor interval computational frequency to coning frequency. [b]Percent error in coning correction $*(-1)$.

where

$$k_1 = 5/5544, \qquad k_2 = 1355/2772, \qquad k_3 = 2955/2772$$

$$k_4 = 3455/2772, \qquad k_5 = 15{,}335/5544$$

Algorithm 9, which is a new algorithm utilizing the structure defined by Eq. (5), incorporates four optimized coefficients in the computation of the coning integral over each minor interval.

Algorithm 10, which is also new, utilizes the enhanced algorithm structure defined by Eq. (7), with sensor data from the last subminor interval of the previous minor interval utilized in the computation of the coning integral over the present minor interval. The algorithm incorporates five optimized coefficients, the greatest number appearing in any algorithm.

Algorithm Accuracies

The accuracies associated with the ten algorithms are defined for a pure coning angular rate environment in Table 1. The accuracies may be ascertained using either of two techniques. The first is to generate a stream of synthetic sensor data for an assumed pure coning angular rate environment, from which the algorithmic approximation to the coning correction over a major computational interval may be computed and compared to the analytically exact value, thereby allowing the error in the algorithmic approximation to be determined. The second method utilizes closed-form expressions for the errors, an example of this approach being provided in the later development.

Reference to the results given in Table 1 shows that in each of the four categories the accuracies associated with the newly defined coning compensation algorithms surpass—in some cases quite significantly—the accuracies associated with the other known algorithms. The substantial accuracy improvements brought about by including sensor data from the previous minor interval in the computation of the coning integral for the present minor interval, as well as by increasing the number of subminor sensor-data intervals used in the computation, will also be noted.

Basic Relationships

As a preliminary to the algorithm derivation of the following section, two basic relationships applicable in a pure coning environment will be developed. These serve as basic building blocks that greatly facilitate the derivation of the various optimized coning compensation algorithms given in the preceding section.

The first basic relationship defines the coning integral over a minor computational interval. This can be derived directly from the defining relationship given by Eq. (3) when the body is undergoing pure coning angular motion, which is characterized by the angular rate vector

$$\omega = a\Omega \cos \Omega\tau \boldsymbol{I} + b\Omega \sin \Omega\tau \boldsymbol{J} \tag{8}$$

The coning integral is evaluated by first integrating ω, as defined, to yield

$$\alpha(\tau) = a \sin \Omega\tau \boldsymbol{I} + b(1 - \cos \Omega\tau)\boldsymbol{J}$$

which, when substituted into Eq. (3), and after a second integration, leads to the exact result for $\delta\boldsymbol{\theta}_c(m)$,

$$\delta\boldsymbol{\theta}_c(m) = (ab\Omega/2)[T - (1/\Omega)\sin \Omega T]\boldsymbol{K} \tag{9}$$

The result given by Eq. (9) reveals the interesting property that the coning integral is constant over all minor intervals, regardless of the absolute time at which the interval begins, but depends only on the duration of the minor computational interval T.

The second basic relationship of interest involves the cross product of two incremental-angle vectors ($\Delta\boldsymbol{\theta}_i \times \Delta\boldsymbol{\theta}_j$) taken over different subminor intervals. Consider first, for the coning angular rate vector defined by Eq. (8), the incremental-angle vector over a subminor interval of duration ΔT ending at t_k

$$\Delta\boldsymbol{\theta}_k = \int_{t_{k-1}}^{t_k} \omega \,\mathrm{d}t = \int_{t_{k-1}}^{t_{k-1}+\Delta T} \Omega(a \cos \Omega t\boldsymbol{I} + b \sin \Omega t\boldsymbol{J})\,\mathrm{d}t$$

$$= a(-U \sin \Omega t_{k-1} + V \cos \Omega t_{k-1})\boldsymbol{I} + b(U \cos \Omega t_{k-1} + V \sin \Omega t_{k-1})\boldsymbol{J}$$

where U and V are defined by

$$U = 1 - \cos\lambda, \qquad V = \sin\lambda$$

in which

$$\lambda = \Omega\Delta T$$

Carrying out the cross product and consolidating terms leads to the desired result

$$\Delta\boldsymbol{\theta}_i \times \Delta\boldsymbol{\theta}_j = ab\{2\sin[(j-i)\lambda] - \sin[(j-i+1)\lambda] - \sin[(j-i-1)\lambda]\}\boldsymbol{K} \tag{10}$$

Like the coning correction, the value of the cross product of two incremental-angle vectors is constant and independent of absolute time but depends only on the duration ΔT of the intervals and their spacing. This result provides the basis for the simplification concept leading to the algorithm structure defined by Eq. (5).

Algorithm Derivation

The algorithm derivation and optimization technique leading to the ten algorithms defined may be illustrated using a specific algorithm as an example. Algorithm 5 will serve this purpose.

The coning integral over the three subminor sensor-data intervals constituting each minor interval is approximated for this algorithm by

$$\delta\hat{\boldsymbol{\theta}}_c(m) = [k_1\Delta\boldsymbol{\theta}_{m-1}(3) + k_2\Delta\boldsymbol{\theta}_m(1) + k_3\Delta\boldsymbol{\theta}_m(2)] \times \Delta\boldsymbol{\theta}_m(3)$$

Assuming a pure coning environment, as defined by Eq. (8), the exact value of the coning integral over the minor interval is determined from Eq. (9) as

$$\delta\boldsymbol{\theta}_c(m) = (ab/2)(3\lambda - \sin 3\lambda)\boldsymbol{K}$$

The individual cross products in the algorithmic approximation are defined for a pure coning angular rate environment by

$$\Delta\boldsymbol{\theta}_{m-1}(3) \times \Delta\boldsymbol{\theta}_m(3) = ab(2\sin 3\lambda - \sin 4\lambda - \sin 2\lambda)\boldsymbol{K}$$

$$\Delta\boldsymbol{\theta}_m(1) \times \Delta\boldsymbol{\theta}_m(3) = ab(2\sin 2\lambda - \sin 3\lambda - \sin\lambda)\boldsymbol{K}$$

$$\Delta\boldsymbol{\theta}_m(2) \times \Delta\boldsymbol{\theta}_m(3) = ab(2\sin\lambda - \sin 2\lambda)\boldsymbol{K}$$

where the expression for the cross product of two incremental-angle vectors given by Eq. (10) was applied to each cross product.

Algorithm optimization is predicated on minimizing the difference between the exact value of the coning integral over a minor interval and its algorithmic approximation; this is accomplished by expanding each expression as a series expansion in odd powers of λ and equating coefficients of like terms up to some maximum number. For Algorithm 5, the coefficients of terms up to and including the seventh power can be equated. This leads to three conditions involving the coefficients k_1, k_2, and k_3 that must be satisfied to yield the optimal set of coefficients, which results in the following set of simultaneous equations:

$$\begin{bmatrix} 3 & 2 & 1 \\ 19 & 6 & 1 \\ 289 & 46 & 3 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 9/4 \\ 81/20 \\ 2187/84 \end{bmatrix}$$

the solution of which provides the three coefficient values

$$k_1 = 3/280, \qquad k_2 = 57/140, \qquad k_3 = 393/280$$

The derivation of all ten algorithms given earlier is accomplished in exactly the same manner, the only difference being in the set of simultaneous conditions that must be satisfied to yield the optimal values of the algorithm coefficients.

Algorithm Error in a Pure Coning Environment

The basic relationships used in deriving the optimal algorithm coefficients may also be employed in establishing the accuracy associated with each algorithm in a pure coning environment. The approach is demonstrated specifically for Algorithm 5, but extension to the general case is straightforward.

The procedure is to first express the error in the algorithmic approximation to the coning integral over a minor interval, using the analytical expression for the vector cross product given earlier, and the optimal coefficients defined for Algorithm 5, as

$$\begin{aligned} \boldsymbol{e}(\lambda) &= \delta\hat{\boldsymbol{\theta}}_c(m) - \delta\boldsymbol{\theta}_c(m) \\ &= ab[(84/35)\sin\lambda - (21/35)\sin 2\lambda + (4/35)\sin 3\lambda \\ &\quad - (3/280)\sin 4\lambda - (3/2)\lambda]\boldsymbol{K} \end{aligned}$$

where $\boldsymbol{e}(\lambda)$ is the error in the algorithmic approximation and λ may be expressed as

$$\lambda = \Omega\Delta T = 2\pi\Delta T/T_c = 2\pi T/3T_c = 2\pi/3r$$

in which T_c is the period associated with the coning angular motion and r is the ratio of the minor interval computational frequency to the coning frequency.

The error incurred over k such intervals is k times the value incurred over a single minor interval, since the latter is constant and depends only on the ratio of the minor interval computational frequency to the coning frequency. It is also true that over the same set of k minor intervals the coning correction has an exact value given by

$$\Delta\boldsymbol{\theta}_c(kT) = (ab/2)(k\Omega T - \sin k\Omega T)\boldsymbol{K}$$

which is obtained by integrating Eq. (1) over the time interval kT. For Algorithm 5, in which each minor interval contains three subminor sensor-data intervals of duration ΔT, the following is true:

$$\begin{aligned} \Delta\boldsymbol{\theta}_c(kT) &= (ab/2)(3k\Omega\Delta T - \sin 3k\Omega\Delta T)\boldsymbol{K} \\ &= (ab/2)(3k\lambda - \sin 3k\lambda)\boldsymbol{K} \end{aligned}$$

If it is assumed, for the purpose of establishing the algorithm error, that the ratio r of the minor interval computational frequency to the coning frequency is a ratio of integer numbers, then a unique time period may be defined that spans both an integer number of minor intervals and an integer number of cycles of the coning motion. This is not a restrictive assumption since, in fact, it is possible to approximate any ratio in this manner with an arbitrarily small error. When the ratio satisfies this constraint, the exact coning correction over the time interval kT that spans both an integer number of minor intervals and an integer number of coning cycles is

$$\Delta\boldsymbol{\theta}_c(kT) = (3abk\lambda/2)\boldsymbol{K}$$

which reflects the fact that the sinusoidal term in the expression for $\Delta\boldsymbol{\theta}_c$ is identically zero for the assumed constraint on the ratio r. Therefore, the percent error e' in the coning correction is defined for Algorithm 5 by

$$e'(\lambda) = 100e(\lambda)/(ab\pi/r)$$

This result, even though derived specifically for Algorithm 5, is the general form applicable to all algorithms, given the appropriate $e(\lambda)$.

Algorithm Error in a Benign Environment

Since the optimization of the algorithms is predicated on the existence of a pure coning angular rate environment, the question arises concerning the extent to which algorithm performance will be compromised in a benign angular rate environment. To illustrate the approach used in determining the error associated with a benign environment, a specific algorithm is evaluated, Algorithm 5 being chosen for this purpose.

Define the benign angular rate environment by the following angular rate equation:

$$\boldsymbol{\omega} = \boldsymbol{A} + \boldsymbol{B}(\tau - \tau_{m-1}) + \boldsymbol{C}(\tau - \tau_{m-1})^2 + \cdots \tag{11}$$

where $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ are vector coefficients and τ is the time measured from the beginning of the minor interval. Integrating Eq. (11) leads to the following expression for $\boldsymbol{\alpha}$:

$$\boldsymbol{\alpha}(\tau, \tau_{m-1}) = \boldsymbol{A}(\tau - \tau_{m-1}) + \tfrac{1}{2}\boldsymbol{B}(\tau - \tau_{m-1})^2 + \tfrac{1}{3}\boldsymbol{C}(\tau - \tau_{m-1})^3 + \cdots \tag{12}$$

Then, from Eqs. (11) and (12), the exact coning integral over the mth minor interval is found to be

$$\begin{aligned} \delta\boldsymbol{\theta}_c(m) &= \frac{1}{2}\int_{\tau_{m-1}}^{\tau_{m-1}+3\Delta T} \boldsymbol{\alpha}(\tau, \tau_{m-1}) \times \boldsymbol{\omega}\,\mathrm{d}\tau \\ &= \frac{9}{4}(\boldsymbol{A} \times \boldsymbol{B})\Delta T^3 + \frac{27}{4}(\boldsymbol{A} \times \boldsymbol{C})\Delta T^4 \\ &\quad + \frac{81}{20}(\boldsymbol{B} \times \boldsymbol{C})\Delta T^5 + \cdots \end{aligned} \tag{13}$$

The incremental-angle vector $\Delta\boldsymbol{\theta}_m(i)$ over the ith subminor interval of the mth minor interval is established in terms of $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ by integrating the angular rate vector over the subminor interval, with the result

$$\Delta\boldsymbol{\theta}_m(1) = \boldsymbol{A}\Delta T + \boldsymbol{B}(\Delta T^2/2) + \boldsymbol{C}(\Delta T^3/3) + \cdots$$

$$\Delta\boldsymbol{\theta}_m(2) = \boldsymbol{A}\Delta T + (3/2)\boldsymbol{B}\Delta T^2 + (7/3)\boldsymbol{C}\Delta T^3 + \cdots$$

$$\Delta\boldsymbol{\theta}_m(3) = \boldsymbol{A}\Delta T + (5/2)\boldsymbol{B}\Delta T^2 + (19/3)\boldsymbol{C}\Delta T^3 + \cdots$$

Likewise, the incremental-angle vector over the last subminor interval of the previous minor interval may be expressed as

$$\Delta\boldsymbol{\theta}_{m-1}(3) = \boldsymbol{A}\Delta T - \tfrac{1}{2}\boldsymbol{B}\Delta T^2 + \tfrac{1}{3}\boldsymbol{C}\Delta T^3 + \cdots$$

Substituting the incremental angle vectors into Algorithm 5, carrying out the indicated cross products, and reducing leads to the

Table 2 Coning algorithm error in a benign environment

Algorithm	Error coefficient[a]
1	1/180
2	1/240
3	1/60
4	1/60
5	16/945
6	29/1701
7	51/2240
8	23/1008
9	83/3150
10	203/7700

[a]Coefficient c in the error expression $e = c(A \times C)T^4 + \cdots$.

algorithmic approximation to the coning integral over the mth computational interval, given by

$$\delta\hat{\theta}_c(m) = (9/4)(A \times B)\Delta T^3 + (1137/140)(A \times C)\Delta T^4 + (339/56)(B \times C)\Delta T^5 + \cdots \tag{14}$$

The error e in the algorithmic approximation is obtained by subtracting the exact expression for the coning integral defined by Eq. (13) from the algorithmic approximation defined by Eq. (14), with the following result

$$e = \delta\hat{\theta}_c(m) - \delta\theta_c(m) = (48/35)(A \times C)\Delta T^4 + (561/280)(B \times C)\Delta T^5 + \cdots$$

or, expressing the error in terms of the minor interval time T, which is equal to $3\Delta T$ for Algorithm 5, and retaining only the first term in the series expansion, leads to

$$e = (16/945)(A \times C)T^4 + \cdots$$

The errors associated with the benign environment are found for the remaining coning algorithms in the same manner, with the result that all ten algorithms may be shown to lead to errors of the same form, with the leading term in the series expansion for the error being proportional to $A \times C$ and the fourth power of the minor interval T. The algorithm errors in a benign environment are summarized by the results given in Table 2.

The relative importance of the benign error to system performance may be assessed by considering the specific example defined by the numerical values $A = 2$ rad/s, $C = 50$ rad/s^3, and $T = 0.005$ s. The resultant attitude errors incurred over the minor interval are defined for the various algorithms in Table 3.

Table 3 Attitude errors during a severe maneuver

Algorithms	Attitude error over a minor interval,[a] rad	Attitude error growth rate,[a] rad/s
1, 2	0.31 $e-9$	0.62 $e-7$
3, 4, 5, 6	1.06 $e-9$	2.12 $e-7$
7, 8	1.43 $e-9$	2.86 $e-7$
9, 10	1.65 $e-9$	3.29 $e-7$

[a]Average value for the algorithms in the group.

The results given in Table 3 may be viewed as worst-case errors for the numerical values assigned to the magnitudes of A and C, since the two vectors are treated as being orthogonal. In all cases the resultant attitude errors may be considered ignorable, even for very high-precision applications, since the growth rates are very small and persist only for the duration of the maneuver.

Conclusions

Six new coning compensation algorithms have been defined, and their accuracy characteristics established for both pure coning and benign angular rate environments. The algorithm enhancement concept defined in the paper was shown to lead to significant improvements in computational efficiency relative to other known algorithms; and its flexibility to incorporate as many subminor sensor-data intervals as required from the previous minor interval, to achieve a desired level of accuracy, was demonstrated. It was further shown that the algorithm errors incurred in a benign environment are negligibly small, even for very severe maneuvers, and that algorithm optimization based strictly on a consideration of its accuracy characteristics in a pure coning environment is justified. The algorithm simplification concept employed in the paper, which is similarly based solely on the properties of the algorithm in a pure coning environment, was also shown to be justified.

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